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POSTCRITICAL REGIMES IN THE NONLINEAR PROBLEM OF VORTEX MOTION UNDER THE FREE SURFACE OF A WEIGHABLE FLUID

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An improved Levi-Civita method in which the singularities of the desired function are taken into account by introducing terms containing power singularities is proposed. Results of numerical analysis of the nonlinear problem of a vortex in a bounded flow of an ideal weighable fluid (Fr > 1) *are given. The following limiting flow regimes are studied: the Stokes waves with one and two crests, emergence of a critical point on the surface, and the detachment of a vortex from a soliton and a uniform flow. It is shown that nonperiodic waves can form in a local zone in the vicinity of the critical point.*

Flow around a hydrofoil can be modeled by flow around a point vortex provided its dimensions are small compared to the distances to the bottom and free surface. This problem was studied in [1–6]. The results obtained in the present study make it possible to describe qualitatively soliton-type solutions for postcritical regimes (Fr > 1). Solutions that can yield the limiting solutions with critical points on the free surface (of the Stokes-wave type with a surface break that forms an angle of 120° or 360° or without a break) are found.

1. Formulation of the Problem. We consider the problem of a vortex of intensity Γ which is located at the point A under the free surface in a bounded flow of an ideal weighable fluid (Fig. 1a). The acceleration of gravity g is directed vertically downward. The velocity is zero at the critical point F (for $\Gamma > 0$, this point lies above the point A), h is the undisturbed thickness of the flow, and V_0 is the velocity at infinity. The flow pattern is symmetric about the y axis. The flow is assumed to be potential and solenoidal; therefore, we seek a solution of the problem in the form of an analytic function of complex variable (complex potential $w(z)$ [6]). In this case, the complex conjugate velocity at any point $z = x + iy$ of the flow is obtained by differentiation $\bar{V} = dw/dz$. To determine $w(z)$, it is necessary to solve a boundary-value problem. The boundary conditions for the function $w(z)$ are the impermeability conditions $\operatorname{Im} w = 0$ on BD and $\operatorname{Im} w = Q$ on CD ($Q = hV_0$ is the fluid flow rate in the jet).

Since the free-surface shape is unknown, a solution can be sought for in the parametric form $w(\zeta)$ and $z(\zeta)$, where ζ is the parametric variable, whose domain is known and can be assumed, for example, to be a semicircle of unit radius in the complex plane (Fig. 1b). The analytic function $z(\zeta)$ is also found from a boundary-value problem in which $y = \operatorname{Im} z = 0$ at the rectilinear boundary BD and the Bernoulli equation

$$(V/V_0)^2 + 2y/(\mathrm{Fr}^2 h) = \mathrm{const} = 1 + 2/\mathrm{Fr}^2, \qquad \mathrm{Fr} = V_0/\sqrt{gh} \tag{1.1}$$

is satisfied at the free surface CD.

The dependence of the complex potential w on ζ can be written in the form [5]

$$w(\zeta) = \frac{2hV_0}{\pi} \ln \frac{1+\zeta}{1-\zeta} + \frac{\Gamma}{2\pi i} \ln \frac{(\zeta - ip)(\zeta + i/p)}{(\zeta + ip)(\zeta - i/p)}. \tag{1.2}$$

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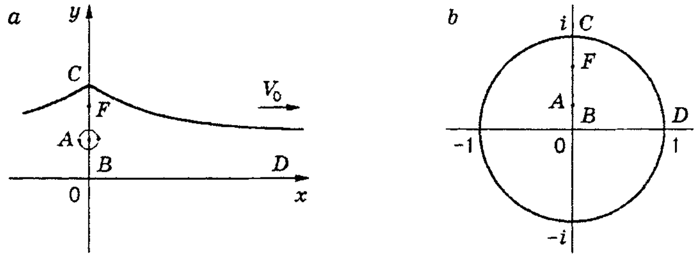

Fig. 1

The complex velocity dw/dz is equal to zero at the critical point F ($\zeta = iq$). Hence, with allowance for (1.2), we obtain

$$\frac{dw}{d\zeta} = \frac{2}{\pi}\Big[\frac{2}{1-\zeta^2} + \frac{\gamma p(1-p^2)}{2}\,\frac{1-\zeta^2}{(\zeta^2+p^2)(p^2\zeta+1)}\Big] = 0, \qquad \gamma = \frac{\Gamma}{hV_0}; \tag{1.3}$$

$$q = \sqrt{\frac{-[2+2p^4-\gamma p(1-p^2)] + 2(1+p^2)\sqrt{1-p^2}\sqrt{1-p^2-\gamma p}}{4p^2+\gamma p(1-p^2)}}. \tag{1.4}$$

2. Method of Direct Conformal Mapping. The problem is solved by the improved Levi-Civita method with allowance for singularities [7, 8].

We now seek the function $z(\zeta)$ in the form of a sum of the power series and certain functions that take into account the specified singularities at the points $\zeta = \pm 1$ and $\zeta = \pm i$:

$$z(\zeta) = (2h/\pi)[z_0(\zeta) + z_1(\zeta) + z_2(\zeta) + z_3(\zeta)], \tag{2.1}$$

where $z_0(\zeta) = \ln((1+\zeta)/(1-\zeta))$ is a function that maps the semicircle onto a band, $z_1(\zeta) = \sum\limits_{m=0}^{\infty} C_{2m+1}\zeta^{2m+1}$ is the power series whose coefficients are determined so as to satisfy the Bernoulli equation (1.1), $z_2(\zeta) = A_1[(1-\zeta)^\alpha - (1+\zeta)^\alpha]$ is a function that takes into account the singularities of the solution $z(\zeta)$ for $\zeta = \pm 1$, and $z_3(\zeta) = iA_2[((1+i\zeta)/2)^{2/3} - ((1-i\zeta)/2)^{2/3}]$ is a function that takes into account the singularities at the break point on the free surface ($\zeta = i$). The parameter α is determined from the transcendental Stokes equation

$$\alpha\frac{\pi}{2}\cot\Big(\alpha\frac{\pi}{2}\Big) = \frac{1}{\mathrm{Fr}^2} \qquad (0 < \alpha < 1). \tag{2.2}$$

3. Solution in Terms of the Joukowski Function. As the studies have shown, the higher-order singularities of the solution must be taken into account in some cases. It is, therefore, convenient to seek a solution in the form of the Joukowski function

$$\omega = \theta + i\tau = i\ln\Big(\frac{1}{V_0}\frac{dw}{dz}\Big), \tag{3.1}$$

where θ is the angle of slope of the velocity vector to the x axis. Equation (1.1) plays the role of a boundary condition to determine the function $\omega(\zeta)$ on the free surface. The following boundary conditions are formulated on the other sections of the boundary: $\theta = 0$ on AB, BD, and FC and $\theta = \pi$ on AF (Fig. 1a). In this case, the conformal mapping of ζ onto z is performed by numerical integration

$$z = \frac{1}{V_0}\int e^{i\omega}\Big(\frac{dw}{d\zeta}\Big)d\zeta. \tag{3.2}$$

We write $\omega(\zeta)$ in the form of the sum

$$\omega(\zeta) = \omega_0(\zeta) + \omega_1(\zeta) + \omega_2(\zeta). \tag{3.3}$$

Here

$$\omega_0(\zeta) = i \ln \frac{(\zeta^2+q^2)(p^2\zeta^2+1)}{(\zeta^2+p^2)(q^2\zeta^2+1)}, \tag{3.4}$$

where $\omega_0(\zeta)$ is the Joukowski function for a similar problem of a weighable fluid [5]);

$$\omega_1(\zeta) = i\,\frac{1-\zeta^2}{2} \sum_{m=0}^{\infty} C_{2m}\zeta^{2m}$$

is the power series into which the factor $(1-\zeta^2)/2$ is introduced to satisfy the condition $\omega_1(1)=0$;

$$\omega_2(\zeta) = iB_1\Big(\frac{1-\zeta^2}{2}\Big)^{\alpha} + iB_2\Big(\frac{1-\zeta^2}{2}\Big)^{2\alpha} + iB_3\Big(\frac{1-\zeta^2}{2}\Big)^{\alpha+1}, \tag{3.5}$$

where $\omega_2(\zeta)$ is a function that takes into account the singularities of the solution $\omega(\zeta)$ for $\zeta = 1$ and involves higher-order terms;

$$\omega_3(\zeta) = \frac{i}{3} \ln \frac{1+\zeta^2}{2} + iC_1\Big[\Big(\frac{1+\zeta^2}{2}\Big)^{\beta} - 1\Big], \tag{3.6}$$

where $\omega_3(\zeta)$ is a function that takes into account the singularities at the break point on the free surface.

Substituting (3.3) into (1.1) and equating terms of the same order as $\zeta \to 1$, we arrive at Eq. (2.2) and determine the coefficients B_2 and B_3 for α:

$$B_2 = B_1^2(3/2)\cot^2(\alpha\pi/2), \qquad B_3 = \alpha B_1. \tag{3.7}$$

Substituting (3.3) into (1.1) for $\zeta \to i$ and equating terms of the same order, we obtain

$$\tau_1\,\frac{\pi}{2} = -C_1 + \sum_{m=1}^{\infty} d_{2m}(-1)^m = \frac{1}{3}\Big[\ln\Big(\frac{3}{\pi}\,\frac{1}{\mathrm{Fr}}\Big) + \ln\Big(1-\frac{\gamma p}{1-p^2}\Big)\Big],$$
$$(\beta+1)(\pi/2)\cot(\beta\pi/2) = 1/\sqrt{3}, \qquad 0<\beta<1. \tag{3.8}$$

4. Numerical Solution. The problem reduces to Eq. (1.1) solved by the collocation method [7, 8]. In the infinite sum z_1 in Eq. (2.1), the number of terms N is finite, and equality (1.1) is satisfied at the discrete points $\sigma_m = \pi m/(2N)$, where $m = \overline{1, N}$. The resulting N nonlinear equations form a system solved by the Newton method with a step chosen [8] with the use of the parameters C_{2m+1}, $m = \overline{0, N-4}$, A_1, A_2, and Fr. This technique is also applied to solve the problem in the case where the function (3.2) is used. The error is estimated according to the Runge rule by comparing the values of the parameters (for example, the Fr number, the coordinates of the point C, etc.) obtained for successively increasing N and also with the use of the maximum residual $\Delta_{\max}$ of Eq. (1.1) calculated at intermediate points between the collocation nodes σ_m.

To verify the developed methods of analysis, we solved the problem of a solitary wave with the Stokes singularity [8–11]. This solution is obtained for $\gamma = 0$.

The following three techniques of solving the problem were compared: a solution is sought for in the form of the function $z(\zeta)$ (2.1) or $\omega(\zeta)$ (3.3) with allowance for only the principal terms of the singularities ($B_2 = B_3 = C_1 = 0$) and with the use of expressions (3.5) and (3.6). The Fr number and other parameters were calculated for $N = 5$–1280. Analysis of the results shows that the maximum residual of Eq. (1.1) can be used to estimate the error in determining the Fr number. In a parametric study, this enables one to avoid lengthy calculations connected with the increase in N at each calculation point.

It follows from the numerical experiments that the first two techniques give close results in determining the Froude number and the error and its rate of decrease as N is twice. The maximum error in calculating the Froude number is $\Delta_{\mathrm{Fr}} \sim 10^{-7}$.

Taking into account the coefficients B_2, B_3, and C_1 and equalities (3.7) and (3.8) decreases the error by a factor of 4 and significantly improves the convergence of the method. In this case, the maximum error in determining the Froude number is estimated to be $\Delta_{\mathrm{Fr}} \sim 10^{-12}$ according to the Runge rule. The results of the solution of the problem by the three techniques coincide up to the errors estimated for each technique.

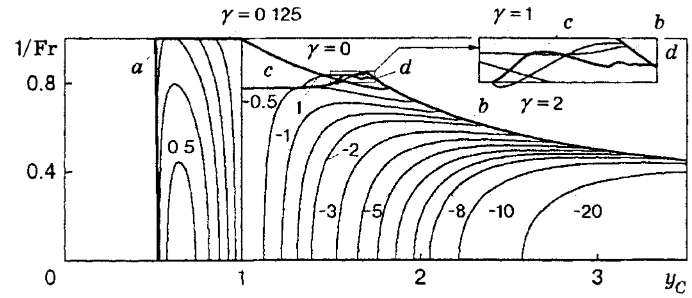

Fig. 2

We compare the refined value of Fr ≈ 1.290890455863, which was obtained by the Richardson scheme, with known results. The values of the Froude number calculated in different studies are Fr ≈ 1.290906 [9], Fr ≈ 1.290889 [10], Fr ≈ 1.29089053 [11], and Fr ≈ 1.290890455 [8]. The errors are $1.6 \cdot 10^{-5}$, $1.5 \cdot 10^{-6}$, $7.5 \cdot 10^{-8}$, and $8 \cdot 10^{-10}$, respectively.

5. Discussion of Results. Figure 2 shows the 1/Fr versus the ordinate y_C of the point C (see Fig. 1) for various vortex intensities γ [$y_C < 1$ for $\gamma = 0.125$, 0.25, 0.375, 0.5, and 0.625, $y_C > 1$ for $\gamma = -0.5$, -1, -1.5, -2, -3, -5, -6, -7, -8, -10, and -20, and $y_C > 1$ (the second solution) for $\gamma = 0$, 1, and 2]. The x and y coordinates in the diagrams are normalized to h. The vortex is located at the height $y_A = 0.5$. For $\gamma > 0$, solutions with a trough (instead of a crest) exist in the region $y_C < 1$. One can see that, for $0 < \gamma < 0.5$, the curve $\gamma = \text{const}$ consists of two curves, one of which emanates from the line Fr = ∞ and ends on the line Fr = 1, and the other emanates from the line Fr = 1 (the second solution) and comes to the curve a, which corresponds to the limiting regime of emerging the critical point F on the free surface. The free-surface shapes are depicted in Fig. 3a (curves 1) for $\gamma = 0.5$ and $y_C = 0.82$ (Fr = ∞), 0.75, 0.65, and 0.51.

Formulas for calculation of the limiting regime of emergence from the critical point F on the free surface can be derived from (1.4) and (3.4) as $q \to 1$:

$$p = \frac{\sqrt{\gamma^2 + 4} - \gamma}{2}, \qquad \omega_0(\zeta) = i \ln \frac{p^2\zeta^2 + 1}{\zeta^2 + p^2}. \tag{5.1}$$

We note that the velocity at the point C is nonzero in this limiting case, i.e., when the critical point reaches the free surface, it disappears. Moreover, the flow corresponds to the two-sheeted Riemann surface, and the velocity vector changes its direction by an angle of 2π in passing over the free surface through the point C. In the particular case, this flow occurs for Fr = ∞. The function ω_0 (5.1) is an exact solution of the problem.

All the solutions considered have no Stokes singularity on the free surface; therefore, in calculations, the terms z_3 and ω_3 are dropped in the corresponding sums (2.1) and (3.3).

The curves $\gamma = \text{const} < 0$ (see Fig. 2; $y_C > 1$) emanate from the line that corresponds to the limiting value of Fr = ∞ (a vortex in a flow of a weightless fluid) and end on the limiting curve that corresponds to a wave with the Stokes singularity (curve b in Fig. 2). The corresponding free-surface shapes are shown in Fig. 3a (curves 2) for $\gamma = -0.5$ and $y_C = 1.12$ (Fr = ∞), 1.3, 1.5, 1.7, 1.9, and 1.95.

The solution $\gamma > 0$ with crests on the free surface originates from the limiting regime of the Stokes-wave type (curve b in Fig. 2) and ends with the Stokes wave with two crests (curves for $\gamma = 1$ and 2 in Fig. 2; Fig. 3b). A limiting curve that corresponds to the regimes with two crests (curve c in Fig. 2) emerges when $\gamma = 0$ ($y_C = 1$) and ends when it adjoins the extreme point of curve b and forms a new non-Stokes limiting regime with the critical point at the top of the crest without a break on the free surface (curve d in Fig. 2). Upon approaching this limit, waves form on the free surface the number of which increases, and their amplitude decreases (Fig. 3c). The value of γ at which this limit is reached is the maximum value for this y_A. When γ decreases, the distance between the crests increases (Fig. 3d). As $\gamma \to 0$, both waves go to infinity, and a

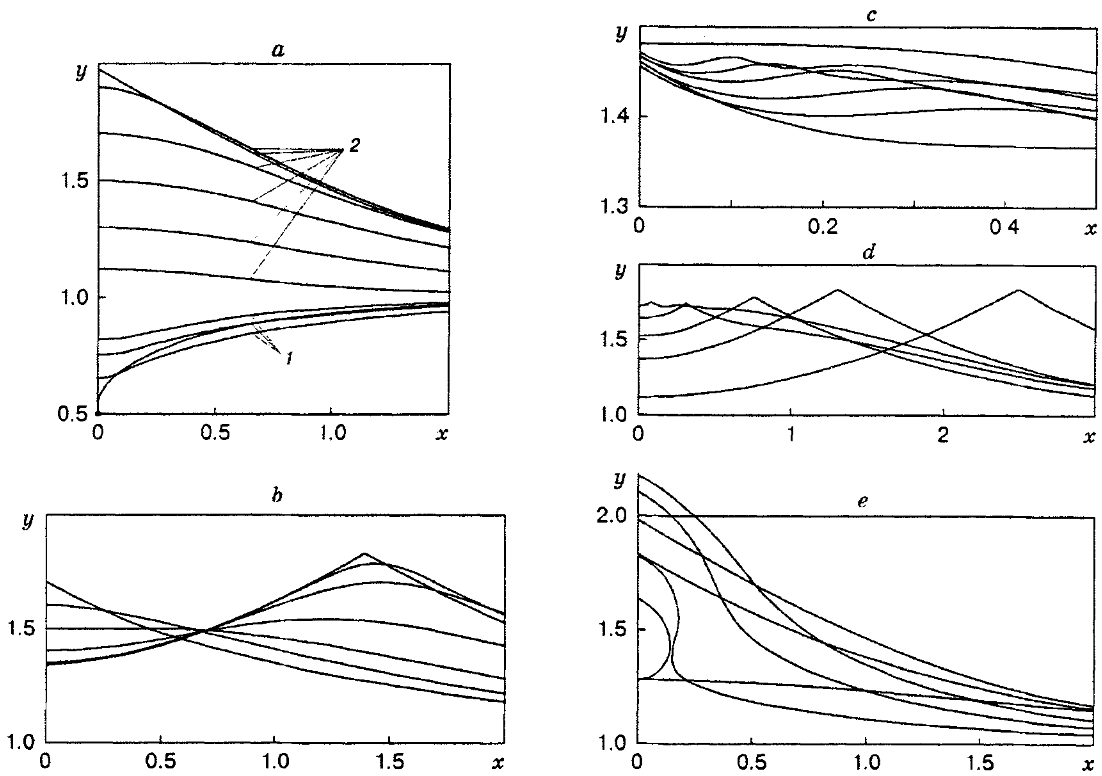

Fig. 3

uniform flow forms between them.

The non-Stokes flow with the critical point at the top of the wave crest is calculated by formula (3.3), in which

$$\omega_0(\zeta) = i \ln \frac{p^2\zeta^2 + 1}{\zeta^2 + p^2} + 2i \ln \frac{1+\zeta^2}{2}, \qquad z_3 = \omega_3 = 0.$$

In contrast to (5.1), the point C is critical, but the free surface remains smooth.

To calculate the flow with two Stokes waves, one should take into account the term ω_3 similar to that in (3.6):

$$\omega_3 = \frac{i}{3} \ln \frac{1 - 2\zeta^2 \cos 2\sigma_0 + \zeta^4}{4 \sin^2 \sigma_0} + iC_1 \Big[\Big(\frac{1 - 2\zeta^2 \cos 2\sigma_0 + \zeta^4}{4 \sin^2 \sigma_0}\Big)^\beta - 1\Big],$$

which has a singularity at the circle point $\zeta = e^{i\sigma_0}$ whose position is determined in the calculation.

The set of solutions of the problem of flow around a vortex is a three-parameter family of curves. To study the solutions in greater detail and show them in the form of a diagram, it is insufficient to use only the section with $y_A = 0.5$ considered above. It is necessary to consider at least the sections with $\gamma =$ const. An analysis shows that, for $\gamma > 0$, the curves $1/\mathrm{Fr}(y_C)$ are similar to those depicted in Fig. 2. Figure 4 shows of the parameters for various y_A corresponding to $\gamma = -0.5$. For $y_A \leqslant 1$, the curves emanate from the line that corresponds to the limiting value of $\mathrm{Fr} = \infty$ and end on the limiting curve of the Stokes-wave type. For a certain $y_A^* > 1$, the curves (for each y_A) consist of two curves (for example, the curve $y_A = 1.25$), one of which emerges from the line corresponding to $\mathrm{Fr} = \infty$ and ends with the Stokes wave as is the case where $y_A \leqslant 1$, whereas the other curve also originates from the line corresponding to $\mathrm{Fr} = \infty$ (the second solution) and ends with the limiting regimes in which the vortex is separated from the soliton (curve *a* in Fig. 4) or uniform flow (curve *b*). When y_A increases, the two solutions for $\mathrm{Fr} = \infty$ merge and disappear, and the two curves coincide (the curve $y_A = 1.5$ in Fig. 4).

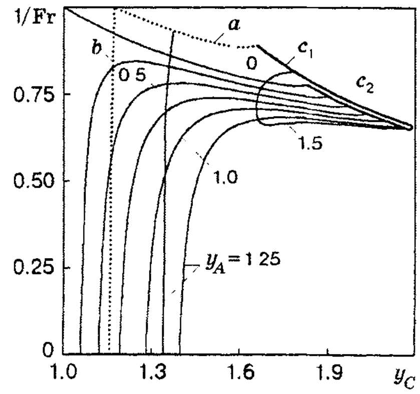

Fig. 4

The free-surface shapes are shown in Fig. 3e for the Stokes-wave type for $\gamma = -0.5$ and $y_C = 1.82$ ($y_A = 0$), 1.98, 2.16, 2.10, 1.823, and 1.63. These solutions correspond to the curve c (a wave with the Stokes singularity) in Fig. 4. It should be noted that $y_C/h = 1 + \mathrm{Fr}^2/2$ for Stokes waves [see (1.1)], but for visualization, the curve c in Fig. 4 is shown in the form of two different curves c_1 and c_2.

We now pass to the problem of vortex separation (see Fig. 4) in which $w = aV_0'\gamma'/(\pi i)\ln\zeta$ is the complex potential, $\gamma' = \Gamma/(aV_0')$, a is the distance from the lower point of the vortex flow to the vortex, and V_0' is the fluid velocity at the lower point. Using (3.3) for $\omega_0(\zeta) = -i\cdot 2\ln\zeta - \pi$ and $\omega_2 = 0$, we find the local value of the Froude number $\mathrm{Fr}' = V_0/\sqrt{ga}$ and obtain

$$\gamma' = \Big(-\frac{1}{\pi}\int\limits_1^0 e^{i\omega(\zeta)}\,\frac{d\zeta}{\zeta}\Big)^{-1}.$$

We note that the problem of closed circulatory flow was solved by Maklakov [12] for sufficiently large Fr'.

Further, using the known solution for a soliton on the free surface [8] in the form of the relationship $\mathrm{Fr}(V_0'/V_0)$, we solve the nonlinear equation

$$(V_0'/V_0)^{3/2}\mathrm{Fr}(V_0'/V_0) = \sqrt{\gamma/\gamma'}\,\mathrm{Fr}', \tag{5.2}$$

thereby finding a relation between the soliton and the vortex flow. It is noteworthy that Eq. (5.2) admits a solution provided $\sqrt{\gamma/\gamma'}\,\mathrm{Fr}' \leqslant 1$. Otherwise, the vortex separates from the uniform flow. In this case, $V_0' = V_0$ and $\mathrm{Fr} = \sqrt{\gamma/\gamma'}\,\mathrm{Fr}'$.

Figures 2 and 4 show solutions only for $\mathrm{Fr} \geqslant 1$. Soliton-type solutions were also obtained by the methods considered above for $\mathrm{Fr} < 1$ (see [3, 4]). However, since Eq. (2.2) has no solutions for $\mathrm{Fr} < 1$, solutions exist only if the equalities $A_1 = 0$ and $B_1 = 0$ hold, i.e., when the power singularities whose index lies in the range $0 < \alpha < 1$ are absent. In this case, the asymptotic behavior of the solution is determined by the next solution of the transcendental equation (2.2) for $2 < \alpha < 3$, which is possible for all values of the Froude number.

6. Conclusions. The problem of a flow of a weighable fluid around a vortex has been studied numerically. The limiting flow regimes have been described. The solutions with waves on the fluid surface for $\mathrm{Fr} > 1$ are of special interest; previously, the characteristic range of Froude numbers for these solutions was assumed to be $\mathrm{Fr} < 1$ (subcritical regimes [4]).

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